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Cauchy-type integrals and integral equations with logarithmic singularities

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SUMMARY

The Gauss-type quadrature methods with a logarithmic weight function can be extended to the evaluation of Cauchy-type integrals and to the solution of Cauchy-type integral equations by reduction of the latter to a linear system of algebraic equations. This system is obtained by applying the integral equation at properly selected collocation points. The poles of the integrand lying in the integration interval were treated as lying outside this interval. The efficiency of the method, both in evaluating integrals and solving integral equations, is exhibited by a numerical example. Finally, an application of the method to a crack problem of plane elasticity is made.

1. Introduction

The problems of the evaluation of the integral

$$I(x) = \int_0^1 \ln\frac{1}{t}\,\frac{\varphi(t)}{t-x}\,dt, \tag{1.1}$$

and the solution of the integral equation

$$\int_0^1 \ln\frac{1}{t}\left\{\frac{K_1(t,x)}{t-x} + K_2(t,x)\right\}\varphi(t)dt = f(x), \tag{1.2}$$

(the solution of a system of integral equations of this type being an obvious generalization), are solved in general by employment of n-point Gaussian quadrature formulae. The kernels $K_1(t, x)$ and $K_2(t, x)$ are bounded in [0,1] with respect to both variables t and x, the unknown function $\varphi(t)$ is supposed continuous in [0,1], and the right-hand side function $f(x)$ is continuous in (0,1] with an appropriate logarithmic singularity at $x = 0$. The coefficients of the appropriate orthogonal polynomials for the logarithmic weight function $\ln(1/t)$, as well as the abscissae and weights of the corresponding Gaussian integration rule, are given by Berthod-Zaborowski [1] for $n = 1(1)4$ and in a much more complete form, appropriate for present-day computations, by Stroud and Secrest [2] for $n = 1(1)16$ and an accuracy of 30 significant digits. An alternative method for the evaluation of these abscissae and weights was proposed by Danloy [3], who considered the more general weight function $w(t) = t^{\alpha}\ln(1/t)$ $(\alpha > -1)$, the numerical results being given for $\alpha = 0$.

Furthermore, Kadlec [4] evaluated integrals with logarithmic singularities and of a more general form than the integral (1.1) for two density functions: $\varphi(t) = 1$ and $\varphi(t) = \exp(-\gamma/t)$ $(\gamma > 0)$. For the first of them he easily derived a closed-form formula by employing the properties of the dilogarithm function $D(x)$ given by Mitchell [5]. For the second density function he reduces the problem to the numerical evaluation of the integral

$$Le(A, B, \gamma) = \int_{1}^{\infty} D(Ax + B)\exp(-\gamma x)dx .$$

A table of values of this integral, accurate to three decimal digits, is given at the end of his paper.

On the other hand, Kulič [6] has also attempted the evaluation of the integral (1.1) in the interval $[-1,1]$ and for any density function, by employing Kantorovich' method of extracting the singularity. In the example given by Kulič the coincidence of the exact and approximate values does not exceed the third significant digit.

Also, Hunter [7] and Chawla and Ramakrishnan [8] modified the Gauss-Legendre and the Gauss-Jacobi methods for the numerical evaluation of the integrals

$$\int_{-1}^{1} \Phi(t)dt \quad \text{and} \quad \int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta}\Phi(t)dt, \quad \alpha, \beta > -1,$$

respectively, to cope with the case where $\Phi(t)$ has simple poles within the interval $(-1,1)$. Such is the case of Cauchy-type integrals, in which

$$\Phi(t) = \frac{\varphi(t)}{t-x},$$

where $\varphi(t)$ is continuous in $[-1,1]$. Furthermore, Theocaris and Ioakimidis have proved, either in [9, 10] following the technique of Hunter [7] and Chawla and Ramakrishnan [8], or in [11, 12] employing the Plemelj formulae, that the treatment of simple poles within the integration interval may be the same as that of simple poles outside this interval, the latter problem having already been studied by Donaldson and Elliott [13]. Thus, the existing quadrature methods have become applicable to the case of Cauchy principal-value integrals. This generalization allows also the solution of systems of Cauchy-type singular integral equations if the roots x_r of the functions $q_n(z)$, given by Eq. (3.7) of [9] are selected as points of application of these equations.

The present solution of the problems of evaluating the integral (1.1) or solving Eq. (1.2) is based on the above-mentioned method of Theocaris and Ioakimidis [9-12]. In Section 2 of this paper the orthogonal polynomials corresponding to the logarithmic weight function are given and the functions $q_n(z)$ are found for the methods of Gauss, Radau and Lobatto. In Section 3 a numerical example, already treated by Kulič [6], is examined both from the point of view of the evaluation of the integral considered there and the solution of the corresponding integral equation. Finally, in Section 4, the method is applied to a plane-elasticity crack problem with a jump in the loading along the crack edges. The strength of the logarithmic singularity in the edge-dislocation density function along the crack at the point of the jump was determined by solving the corresponding Cauchy-type singular integral equation by the method proposed in this paper. The results obtained are seen to converge to their expected value in this simple application.

2. The logarithmic weight function

For $w(t) = \ln(1/t)|[0, 1]$, the orthogonal polynomials are given by the recurrence formula [2, p. 90]:

$$P_0(z) = 1,\ P_1(z) = z - b_1,$$
$$P_n(z) = (z - b_n)P_{n-1}(z) - c_n P_{n-2}(z), \quad n \geq 2. \tag{2.1}$$

In the same reference the coefficients b_n and c_n are given on pp. 90-91 and the roots of the above-mentioned polynomials, as well as the corresponding weights for the case of the Gauss rule in Table 9 (pp. 301-304) for $n = 1(1)16$.

Thus, according to Eq. (3.7) of Ref. [9] and Eq. (2.1) above, the functions $q_n(z)$ for the case under consideration, denoted for the system of polynomials $P_n(z)$ by $Q_n(z)$, are given by

$$Q_n(z) = -\frac{1}{2}\int_0^1 \ln\frac{1}{t}\,\frac{P_n(t)}{t-z}\,dt,$$

and, because of the last of Eqs. (2.1), one obtains

$$Q_n(z) = (z - b_n)Q_{n-1}(z) - c_n Q_{n-2}(z), \quad n \geq 2. \tag{2.2}$$

For $n = 1$, Eq. (2.2) takes the form

$$Q_1(z) = (z - b_1)Q_0(z) - 1/2. \tag{2.3}$$

Finally, in accordance with the definition of the principal value of a Cauchy-type integral, one obtains for the function $Q_0(z)$ in the interval (0,1):

$$Q_0(x) = \frac{1}{2}\lim_{\epsilon\to 0}\left\{\int_0^{x-\epsilon}\frac{\ln t}{t-x}\,dt + \int_{x+\epsilon}^1 \frac{\ln t}{t-x}\,dt\right\}, \quad 0 < x < 1.$$

Hence

$$2Q_0(x) = \ln|x|\ln\left|\frac{1-x}{x}\right| - D\left(\frac{x-1}{x}\right) + D(1) - \lim_{\epsilon\to 0}\left\{D\left(\frac{\epsilon}{x}\right) - D\left(-\frac{\epsilon}{x}\right)\right\},$$
$$0 < x < 1, \tag{2.4}$$

where

$$D(z) = -\int_0^z \frac{\ln(1-u)}{u}\,du$$

denotes the dilogarithm function [5].

Now, since the power-series expansion of the dilogarithm function $D(z)=\sum_{n=1}^{\infty}(z^n/n^2)$ converges absolutely and uniformly for $|z|<1$, [5], the limit in Eq. (2.4) is equal to zero. Thus, Eq. (2.4) takes the simpler form:

$$2Q_0(x)=\ln|x|\ln\left|\frac{1-x}{x}\right|-D\left(\frac{x-1}{x}\right)+D(1),$$

which holds also for the case where $x<0$ or $x>1$, [4, Eq. 11]. In this way, it has become possible to express the function $Q_0(x)$, which is a Cauchy-type integral, interpreted in the principal-value sense for $0<x<1$, through well-defined functions and not through another principal-value integral. Of course this fact is quite unusual. Next, by replacing $D(1)$ and $D((x-1)/x)$ by their expressions given by Eqs. (2.3) and (5.4) of [5] respectively, we obtain

$$2Q_0(x)=\begin{cases}-D(x)-\dfrac{1}{2}\ln^2|x|+\dfrac{\pi^2}{3}, & \text{for } x>0,\\[2mm] -D(x)-\dfrac{1}{2}\ln^2|x|-\dfrac{\pi^2}{6}, & \text{for } x<0.\end{cases} \tag{2.5a}$$

It can also be seen that the function $Q_0(z)$ has, in the case of complex z, the expression

$$2Q_0(z)=-D(z)-\frac{1}{2}\ln^2(-z)-\frac{\pi^2}{6}. \tag{2.5b}$$

The recurrence relation (2.2), together with Eqs. (2.3) and (2.5), completely define the functions $Q_n(z)$ in the whole plane. These functions, together with the polynomials $P_n(z)$, given by Eqs. (2.1), enable the application of the methods of Gauss, Radau and Lobatto:

i) to the evaluation of Cauchy-type integrals with a logarithmic singularity, and

ii) to the solution of Cauchy-type singular integral equations with a logarithmic singularity.

Actually, the polynomials $P_n(z)$ and the corresponding functions $Q_n(z)$ are used only for the Gauss method. In the Radau method the appropriate polynomials are given by [9]

$$\sigma_n(z)=P_n(z)+d_nP_{n-1}(z),\quad n\geqslant 1, \tag{2.6}$$

where

$$d_n=-\frac{P_n(\tau)}{P_{n-1}(\tau)},\quad \tau=0\text{ or }1.$$

Furthermore, in the Lobatto method the appropriate polynomials are given by [9]:

$$\sigma_n(z)=P_n(z)+d_nP_{n-1}(z)+e_nP_{n-2}(z),\quad n\geqslant 2, \tag{2.7}$$

where

$$\begin{aligned}d_n&=-\left\{P_n(0)P_{n-2}(1)-P_n(1)P_{n-2}(0)\right\}/D,\\ e_n&=\left\{P_n(0)P_{n-1}(1)-P_n(1)P_{n-1}(0)\right\}/D,\\ D&=P_{n-1}(0)P_{n-2}(1)-P_{n-1}(1)P_{n-2}(0).\end{aligned}$$

It can be readily seen that the functions $q_n(z)$, in the cases of the Radau and Lobatto methods, are also given by Eqs. (2.6) and (2.7) respectively, provided that $P_n(z), P_{n-1}(z)$ and $P_{n-2}(z)$ are replaced by $Q_n(z), Q_{n-1}(z)$ and $Q_{n-2}(z)$ respectively. Hence, the weights for either of the three above-mentioned methods of n-point quadrate rules are [9]

$$A_k = -2q_n(t_k)/\sigma_n'(t_k), \quad k = 1, 2, ..., n, \tag{2.8}$$

where $\sigma_n(z)$ and $q_n(z)$ denote the functions corresponding to the method employed and t_k are the roots of $\sigma_n(z)$.

Thus, for any value of x, other than 0, 1 and t_k $(k = 1, 2, ..., n)$, the integral (1.1) can be evaluated from the well-known quadrature formula [9]:

$$I(x) = \sum_{k=1}^{n} A_k \Phi(t_k) - 2 \sum_{k=1}^{m} \frac{\rho_k q_n(z_k)}{\sigma_n(z_k)} + E_n, \tag{2.9}$$

where

$$\Phi(t) = \frac{\varphi(t)}{t - x},$$

z_k are the simple poles of $\Phi(z)$ lying in the interior of a simple closed curve C surrounding the integration interval [0,1] and ρ_k are the residua of $\Phi(t)$ at these poles. As regards the error term E_n, it is given by [9]:

$$E_n = \frac{1}{\pi i} \int_C \frac{q_n(\tau)}{\sigma_n(\tau)} \Phi(\tau) d\tau.$$

Now, for $0 < x < 1$ one of the poles, namely $t = x$, lies within the integration interval and the corresponding term in the second sum of Eq. (2.9) must be taken into consideration unless x is a root of $q_n(z)$. Furthermore, it should be noted that the influence of other poles lying in the vicinity of the integration interval may be significant and this is exhibited in the numerical example treated in Section 3.

On the other hand, when Eq. (2.9) is employed for the solution of the integral equations, the roots of $q_n(z)$ in the integration interval are selected as the points x_r of its application, so that there remain in the second sum of its right-hand side only terms corresponding to poles outside the integration interval. Although these terms are ignored at present, their influence on the resulting values of $\varphi(t_k)$ becomes considerable for those t_k lying in the vicinity of such poles.

3. A numerical example

Kulič [6] has evaluated the integral

$$J(x, a) = \frac{1}{\pi} \int_{-1}^{1} \frac{\ln |t|}{t^2 + \tan^2 a} \cdot \frac{dt}{t - x}, \quad -1 \leqslant x \leqslant 1, \tag{3.1}$$

as an application of his method developed in the same paper. The exact value of this integral for $a = \pi/4$ is

$$\bar{J}(x, \pi/4) = \{2xG + (\pi^2/2)\,\mathrm{sign}\,x - N(x)\} \,/\, \{\pi(1 + x^2)\}, \tag{3.2}$$

where $G = 0.915965594177219015$ is Catalan's constant and

$$N(x) = \int_0^x \ln \frac{1+\tau}{1-\tau}\frac{d\tau}{\tau} = D(x) - D(-x), \tag{3.3}$$

where $D(x)$ is the dilogarithm function.

This example is also treated here both as a problem of evaluating the integral (3.1) for $a = \pi/4$ and as one of solving the integral equation

$$J(x, \pi/4) = \bar{J}(x, \pi/4).$$

To this end, the integration interval $[-1, 1]$ must be transformed into $[0, 1]$:

$$J(x, \pi/4) = \frac{1}{\pi}\int_0^1 \ln|t|\,K(t, x)\varphi(t)dt \tag{3.4}$$

where the kernel $K(t, x)$ is given by

$$K(t, x) = \left(\frac{1}{t-x} - \frac{1}{t+x}\right)\frac{1}{1+t^2}, \quad 0 \leqslant x \leqslant 1, \tag{3.5}$$

and

$$\varphi(t) = 1.$$

As regards the dilogarithm function $D(x)$, it was computed by using the algorithm of Ginsberg and Zaborowski [14], valid along the whole real axis with an accuracy of about 16 digits.

3.1 *Evaluation of the integral*

From Eq. (3.5) one can see that the poles to be considered in evaluating the integral (3.4) are $t = x$ (lying inside the integration interval) and $t = -x$ (lying outside this interval; the simple poles $t = \pm i$, lying for any value of x, $0 \leqslant x \leqslant 1$, far from the interval [0,1] will not be taken into account. The terms in the second sum in Eq. (2.9) corresponding to these poles are

$$r(x) = \frac{1}{1+x^2}\cdot\frac{q_n(x)}{\sigma_n(x)}, \quad r(-x) = -\frac{1}{1+x^2}\cdot\frac{q_n(-x)}{\sigma_n(-x)}, \tag{3.6}$$

TABLE 1

Comparison of the numerical results obtained for the integral $J(x, \pi/4)$ by using the Gauss method (columns 2 and 3) or the method of Kulič (column 5) with its exact values (column 4).

Column 1 x	Column 2	Column 3	Column 4	Column 5
0.00001	−12.0195219613054	1.57079579165814	1.57079579165817	1.570796
0.0001	−4.54365692615055	1.57079096129050	1.57079096129049	1.570791
0.001	0.40547264089362	1.57074125801857	1.57074125801860	1.570740
0.01	1.56102926022973	1.57010426598715	1.57010426598716	1.570092
0.1	1.54987676743279	1.54987677116726	1.54987677116732	1.549752
0.2	1.49954089473264	1.49954089473350	1.49954089473350	1.499282
0.4	1.33153940641463	1.33153940641463	1.33153940641463	1.330916
0.6	1.11836011939027	1.11836011939027	1.11836011939030	1.116911
0.8	0.90170416255557	0.90170416255557	0.90170416255557	0.897479
1.0	0.68425998572954	0.68425998572954	0.68425998572954	0.693580

respectively. Now, if the integral is evaluated at the roots x_r of $q_n(z)$, then $r(x_r) = 0$. For $x \neq x_r$ the correction term $r(x)$ must be considered. On the other hand, the influence of the pole $t = -x$ is rapidly diminishing as x moves away from 0, but it is significant when x is near 0, because, in this case, the pole lies in the vicinity of the integration interval.

The values of the integral $J(x, \pi/4)$ were evaluated by using the Gauss method with $n = 16$ abscissae for different values of x, and they are given in column 2 of Table 1 in the case where the term $r(-x)$ has been ignored, and in column 3 of the same Table in the case where this term has been taken into consideration. The much better approximate value of column 3 for small values of x exhibit the above-mentioned influence of the pole $t = -x$. The exact values $\bar{J}(x, \pi/4)$ of $J(x, \pi/4)$ are given in column 4 of Table 1 and the approximate values, calculated in accordance with the method of Kulič [6] and with $n = 16$ terms, in column 5. It may also be noted that analogous results have been obtained by using the Radau or Lobatto methods instead of the Gauss method. Furthermore, no considerable improvement of the numerical results has been observed when using the method of Kulič for increased values of the number n of terms taken into account.

3.2 *Solution of the integral equation*

Now we will try to solve the integral equation

$$J(x, \pi/4) = \bar{J}(x, \pi/4) \tag{3.7}$$

by reduction to a system of linear equations. At first, we approximate the integral $J(x, \pi/4)$, by using the Gauss, Radau or Lobatto method of numerical integration with n abscissae, by

$$J(x, \pi/4) \simeq \sum_{k=1}^{n} A_k K(t_k, x)\varphi(t_k) - 2\{r(x) + r(-x)\} \tag{3.8}$$

with $r(x)$ and $r(-x)$ given by Eqs. (3.6) and $\varphi(t_k)$ being the n values of the 'unknown' function $\varphi(t)$ at the abscissae t_k. To avoid insertion of the additional unknows $\varphi(x)$ and $\varphi(-x)$ through $r(x)$ and $r(-x)$ respectively, the following procedure should be followed:

i) equation (3.7) must be applied at the roots x_r of $q_n(x)$, so that $r(x_r) = 0$, and
ii) term $r(-x)$ must be ommitted.

Now it has been found numerically that the number of roots x_r of $q_n(x)$ is $(n + 1)$, n and $(n - 1)$ for the Gauss, Radau and Lobatto methods respectively. For each method these roots x_r alternate with the abscissae t_k used. It should also be observed that, although the numerical evaluation of the roots x_r for $n = 1(1)16$ is adequate for the needs of the present research, that is the necessary points of application of the integral equation have been made available, still a theoretical investigation of the exact number of the roots x_r for any value of n would be interesting.

In addition, it has been found that, for all values of $n = 1(1)16$, the smallest root x_1 in either the Gauss method or the Radau method with $t_n = 1$ lies very close to 0, so that the whole solution is significantly influenced by the pole $t = -x_1$ in the numerical example considered here. To overcome the difficulty posed by the fact that the corresponding correction term $r(-x)$ must be omitted, as has been explained previously, the following steps should be taken:

i) In the case of the Gauss method, the smallest root x_1 of $q_n(x)$ is not used as a collocation point.
ii) In the case of the Radau method with $t_n = 1$, the first root x_1 is also omitted and the largest root x_{n+1} of the Gauss method is chosen as the n-th collocation point.
iii) Also in the case of the Lobatto method, the largest root x_{n+1} of the Gauss method is used as the n-th collocation point.

TABLE 2

Numerical results for the values $\varphi(t_k)$ of the unknown function $\varphi(t)$, obtained from the solution of Eq. (3.7) by using the Gauss, Radau and Lobatto methods with $n = 16$ abscissae.

Gauss method	Radau method ($t_1 = 0$)	Radau method ($t_n = 1$)	Lobatto method ($t_1 = 0$ and $t_n = 1$)
1.000120	0.932184	1.000753	0.931770
0.999485	1.012699	0.998791	1.012186
0.999914	1.001343	0.998990	1.000536
0.999969	1.000383	0.998764	0.999298
0.999985	1.000169	0.998405	0.998734
0.999990	1.000096	0.997878	0.998173
0.999993	1.000065	0.997074	0.997411
0.999994	1.000050	0.995788	0.996229
0.999995	1.000043	0.993625	0.994260
0.999995	1.000041	0.989777	0.990755
0.999995	1.000041	0.982417	0.984035
0.999994	1.000045	0.966847	0.969781
0.999993	1.000052	0.928529	0.934621
0.999992	1.000065	0.807407	0.823289
0.999989	1.000092	0.178774	0.244815
0.999980	1.000161	−31.157624	−28.612200

Numerical results, for the values of the unknown function $\varphi(t)$ at the abscissae t_k used, are given in Table 2 for all four methods used with $n = 16$ abscissae. These results, compared with the exact value of the unknown function $\varphi(t) = 1$, appear to be satisfactory.

4. An application

In this section we will apply the results of the previous sections to a plane elasticity problem, namely the problem of a simple straight crack in an infinite isotropic elastic medium in the case when the loading along the crack edges presents a jump. In general, logarithmic singularities are often encountered in plane or antiplane elasticity problems and the most common case when they appear is the case when jumps in loading along the boundary of a medium are present. This becomes clear from the complex-variable formulation of plane elasticity problems contained in the well-known book of Muskhelishvili [15]. An analogous formulation is valid for antiplane elasticity problems. In practice, the experimental method of caustics has been used [16] in plane elasticity problems near points of the boundary of the medium where jumps in loading were present and, hence, the complex potential $\Phi(z)$ [15] presented logarithmic singularities.

In this section we will consider just a simple example of the application of the results of this paper in plane elasticity. Thus we consider the Cauchy-type singular integral equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(t)}{t-x}dt = f(x), \quad -1 < x < 1, \tag{4.1}$$

accompanied by the condition

$$\int_{-1}^{1}\varphi(t)dt = 0. \tag{4.2}$$

These equations arise in the problem of a simple straight crack of length equal to 2 inside an infinite isotropic elastic medium [17]. In Eq. (4.1) $f(x)$ is the loading distribution along the crack edges (assumed the same on both edges) and $\varphi(x)$ an unknown density function, which may be interpreted as an edge-dislocation density along the crack. As regards Eq. (4.2), it assures the single-valuedness of displacements around the crack.

In the case when $f(x)$ is a continuous function along the crack $[-1,1]$, the results of [17] are directly applicable to the solution of Eqs. (4.1-4.2). In our case we will assume that $f(x)$ is discontinuous at some point along the crack. Loading discontinuities often arise in practical applications. For example, in the special problem under consideration, discontinuities in $f(x)$ become automatically present if we consider the well-known Dugdale-Barenblatt model [18]. Here, for simplicity, we will assume that the loading-distribution function $f(x)$ presents a jump at $x = 0$. For the sake of simplicity we assume that

$$f(x) = \operatorname{sign} x. \tag{4.3}$$

More complicated cases can also be treated in a completely analogous way.

By taking into account Eqs. (4.1) and (4.3), we can easily remark that the unknown function $\varphi(t)$ in Eq. (4.1) is an even function. Moreover, because of the expected logarithmic singularity of $\varphi(t)$ at $t = 0$, we replace it by the function:

$$g(t) = \varphi(t)/w(t), \quad w(t) = \ln(1/|t|). \tag{4.4}$$

Now Eqs. (4.1-4.2) can be written as:

$$\frac{2x}{\pi}\int_0^1 \ln\frac{1}{t}\, g(t)\frac{1}{t^2 - x^2}\,dt = f(x), \quad 0 \leqslant x < 1, \tag{4.5}$$

$$\int_0^1 \ln\frac{1}{t}\, g(t)dt = 0. \tag{4.6}$$

Equation (4.5) was solved by using the method of this paper. The Radau rule with an abscissa at $t = 0$ was preferred since it permits the direct evaluation of the intensity of the logarithmic singularity of $\varphi(t)$ at $t = 0$, that is the value of $g(0)$. The number n of abscissae used was $n = 4(4)16$. The collocation points used were $(n - 1)$ in number. The collocation point near the point $x = 1$ was not used; condition (4.6) was used as the nth equation of the system of linear equations after the Radau numerical integration rule was applied to this condition too.

TABLE 3

Convergence of the numerical results for the intensity of the logarithmic singularity at t = 0 in the crack problem of Section 4 to the expected value.

n	4	8	12	16	Theoretical value
$g(0)$	−0.582	−0.602	−0.607	−0.610	$-2/\pi = -0.6366$

In Table 3 we give the values of $g(0)$ obtained for $n = 4(4)16$. We see from this table that the numerically obtained results for $g(0)$ converge to the expected value which is equal to $-2/\pi$. This fact can easily be established if we take into account that the solution of Eq. (4.1) is of the form:

$$\varphi(x) = -\frac{1}{\pi(1 - x^2)^{1/2}}\int_{-1}^1 \frac{(1 - t^2)^{1/2} f(t)}{t - x}\,dt + \frac{C}{(1 - x^2)^{1/2}}, \tag{4.7}$$

where C is a constant to be determined from Eq. (4.2). Of course, a closed-form evaluation of the integral in Eq. (4.7) is not always a simple matter. Yet, in the example of this section, where $f(x)$ is determined from Eq. (4.3), this is possible. Thus, by taking into account that

$$\int_0^1 \frac{(1 - t^2)^{1/2} d(t^2)}{t^2 - x^2} = -2 + 2(1 - x^2)^{1/2}\tanh^{-1}(1 - x^2)^{1/2}, \tag{4.8}$$

we can conclude that

$$\varphi(x) = -\frac{2}{\pi}\tanh^{-1}(1-x^2)^{1/2} + D(1-x^2)^{-1/2}, \tag{4.9}$$

where D is a constant, and, further, that

$$g(0) = -2/\pi, \tag{4.10}$$

Eq. (4.4) also being taken into account.

Finally, we can mention the generalization of these results to more complicated cases. In a general plane elasticity problem, we have usually to split its boundary into parts and use appropriate numerical integration rules in each one of them. For example, we have to use numerical integration rules associated with a logarithmic weight function about points where logarithmic singularities are present (e.g. points of jump in loading), numerical integration rules associated with a weight function presenting a power singularity of order $(-1/2)$ near crack tips, etc. Particularly, in the present example we could have split the integration interval into two parts $[0, 1/2]$ and $[1/2, 1]$ and used a weight function with a logarithmic singularity at $t = 0$ (as really made) in the first subinterval and a weight function with $(-1/2)$ power singularity (at $t = 1$) in the second subinterval. That was not made here since we focussed our attention on the point $t = 0$, where the logarithmic singularity was present, not at the crack tips. Of course, more rapid convergence of the numerical results of Table 3 to their theoretical value would be expected if the inverse-square-root power singularity at the crack tips ($t = \pm 1$) were taken into account. Inversely, we could have ignored the logarithmic singularity at $t = 0$ and used the weight function

$$w(t) = (1 - t^2)^{-1/2} \tag{4.11}$$

along the whole crack $[-1,1]$ or simply $[0,1]$. Then the values of the stress-intensity factor at the crack tips would easily result, but not the intensity of the logarithmic singularity at $t = 0$, contrary to what was made previously.

5. Conclusions

Until now the attempts made in the area of numerical evaluation of Cauchy-type integrals with a logarithmic weight function concern only the evaluation of such integrals either for any function, with moderate accuracy (e.g. Kulič [6]), or for some specific functions (e.g. Kadlec [4]). In this paper a method of numerical integration, recently developed and treating poles inside the integration interval in the same manner as those lying outside it, is adopted to treat logarithmic singularities in the density function of the Cauchy-type integral. The method permits both the evaluation of such integrals with excellent accuracy and the solution of integral equations of the same kind with very good accuracy. The results of this paper are of particular significance in plane and antiplane elasticity problems, where Cauchy-type integrals and integral equations with logarithmic singularities are frequently encountered.

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